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On the Question of a Cosmological Rest-mass of Gravitons

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In many papers (Tonnelat, 1965) authors assert that Einstein's vacuum equations with the cosmological term,

$$R_{kl} + \lambda g_{kl} = 0 \tag{1}$$

give gravitons with a very small but finite rest-mass μ. According to these papers, the square of the rest-mass of gravitons is given by the cosmological constant†

$$\lambda \sim \frac{\mu^2 c^2}{h^2} = k^2 \tag{2}$$

In some papers (Treder, 1963) I have given arguments for a contrary viewpoint: I have asserted that Einstein's gravitation equations with the cosmological term define gravitons with the rest-mass zero as the ordinary Einstein equations do. I will give a general proof for my point in this paper.

In my early papers I discussed the propagation of infinitesimal perturbations γ_{kl} of the gravitational field for a flat Minkowskian space. The metric tensor is, with the Minkowski-tensor η_{kl},

$$g_{kl} = \eta_{kl} + \gamma_{kl} \qquad \text{and} \qquad |\gamma_{kl}| \ll 1 \tag{3}$$

The unphysical vector gravitons are excluded by the Hilbert coordinate conditions,

$$(\gamma_i^k - \tfrac{1}{2}\delta_i^k \gamma), \qquad k = 0 \quad (\text{with } \gamma = \eta^{kl}\gamma_{kl}) \tag{4}$$

By (3) and (4) the linearized Einstein equations (1) are,

$$\tfrac{1}{2}\eta^{mn}\gamma_{kl,\,mn} + \lambda(\eta_{kl} + \gamma_{kl}) = 0 \tag{5}$$

This means that for the γ_{kl} we do not find homogeneous equations of the Klein-Gordon-Yukawa type,

$$\eta^{mn}\gamma_{kl,\,mn} + k^2\gamma_{kl} = 0 \tag{6}$$

† P. G. Bergmann promotes this opinion in his paper on the scalar-tensor theory of gravitation (Bergmann, 1968).

Indeed, the exact spherical symmetric solution of the cosmological Einstein equation (1) is (in Schwarzschild polar coordinates),

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{\lambda}{3} r^2\right)^{-1} dr^2 - r^2 \, d\Omega^2 + \left(1 - \frac{2m}{r} + \frac{\lambda}{3} r^2\right) dt^2 \quad (7)$$

This Weyl-Trefftz solution (7) is the only spherical symmetric solution of (1) according to Birkhoff's theorem. The metric (7) is a first approximation solution of the linearized field equations,

$$\Delta \gamma_{kl} = 2\lambda \eta_{kl} \quad (8)$$

equation (8) corresponds to the inhomogeneous equation (5) and does not correspond to the Yukawa-type equation (6).

The authors who interpret the cosmological constant λ like the square of the rest-mass of gravitons according to (2), put forward as a general argument for their opinion that the variation of the cosmological equation (1) gives Yukawa-type equations for the perturbations of the gravitational field.

The metric g_{kl} defined by (1) may be perturbed. We are able to describe these perturbations by variations δg_{kl} of the background metric g_{kl},

$$g_{kl} \rightarrow g_{kl} + \delta g_{kl} = g_{kl} + \gamma_{kl} \quad (9)$$

The variation (9) of the metric gives a variation of the Christoffel symbols Γ^i_{kl} according to Palatini's formula,

$$\delta \Gamma^i_{kl} = \tfrac{1}{2} g^{ir}(-\gamma_{kl;r} + \gamma_{lr;k} + \gamma_{rk;l}) \quad (10)$$

(The covariant derivatives are defined in the background metric g_{kl}).

From (10) a variation of the Ricci tensor results,

$$\delta R_{kl} = -\delta \Gamma^i_{kl;i} + \delta \Gamma^i_{ki;l} \quad (11)$$

From (1) the perturbed Einstein equations are

$$R_{kl} + \delta R_{kl} + \lambda(g_{kl} + \delta g_{lk}) = 0 \quad (12)$$

The background metric g_{kl} fulfils the equation (1). Therefore, the equations

$$\delta R_{kl} + \lambda \delta g_{kl} = 0 \quad (13)$$

result for the propagation of the perturbations $\delta g_{kl} = \gamma_{kl}$. The equations (13) are apparently Yukawa-type equations. However, from (13) the equations

$$\tfrac{1}{2}(\gamma^s_{kl;s} + \gamma_{;kl} - \gamma^s_{k;ls} - \gamma^s_{l;ks}) + \lambda \gamma_{kl} = 0 \quad (14)$$

are obtained according to (10) and (11) (with $\gamma = g^{mn}\gamma_{mn}$) (Treder, 1962). The vector gravitons are excluded by the covariant generalization of the Hilbert conditions,

$$(\gamma_k{}^l - \tfrac{1}{2}\delta_k{}^l\gamma)_{;l} = 0 \tag{15}$$

Equation (14) leads to the conditions (15)

$$\gamma^s_{kl;s} - 2R_{mkln}\gamma^{mn} + R_k{}^n\gamma_{ln} + R_l{}^n\gamma_{kn} + 2\lambda\gamma_{kl} = 0 \tag{16}$$

This means that we obtain propagation equations, including terms with the Riemannian tensor and terms with the Ricci tensor of the background metric.

From the Einstein equations without the cosmological term $R_{kl} = 0$ the equation

$$\gamma^s_{kl;s} - 2R_{mkln}\gamma^{mn} = 0 \tag{17}$$

result. But, if the background metric fulfils the cosmological equations (1), we have

$$R_k{}^n\gamma_{ln} + R_l{}^n\gamma_{kn} = -2\lambda\gamma_{kl} \tag{18}$$

Therefore, the terms of (16) with the cosmological constant λ are compensated for by the terms with the Ricci tensor.

The result is that the same propagation equations (17) for the perturbations δg_{kl} result from the cosmological equations (1) as from the equations

$$R_{kl} = 0$$

This means that the final form of the propagation equations for the perturbations of the gravitational field is independent of the existence of a cosmological term in the Einstein vacuum equations. Therefore, the gravitons connected with these perturbations have zero rest-mass for a cosmological constant $\lambda \neq 0$ too. The cosmological constant does not have any connection with a graviton rest-mass.

References

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